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NOTE ON "AN APPLICATION OF ELLIPTIC FUNCTIONS TO GEOMETRY." *

By Prof. Jas. H. Boyd, St. Paul, Minn.

Formulas (6) and (11), Annals of Mathematics, Vol. VI, pp. 95 and 97, give

$$S_A(z) = \sum_{+\infty}^{-\infty} {}^n \frac{\pi \tan^4 \alpha \,.\, h^{4n} z^4}{(h^{2n} z^2 - z_1^2)^2 (h^{2n} z^2 - z_2^2)^2} \tag{1}$$

$$= C_1 \, \mathrm{p}\,(u - u_1) + C_2 \, \mathrm{p}\,(u - u_2) + C_3 \frac{\sigma'}{\sigma}(u - u_1) + C_4 \frac{\sigma'}{\sigma}(u - u_2) + C_5 \,,$$

where $S_A(z)$ represents the sum of the areas of circles which touch each other successively and are inscribed in a crescent formed by the intersection of two circles. The C's are constants to be determined.

We know by the theory of decomposition of fractions into their partial fractions, that,

$$\frac{x}{(x-\alpha)^2 (x-\beta)^2} = \frac{A_0}{(x-\alpha)^2} + \frac{A_1}{x-\alpha} + \frac{B_0}{(x-\beta)^2} + \frac{B_1}{x-\beta}\,,$$

where,

$$A_0 = \frac{\alpha}{(\alpha-\beta)^2}\,, \qquad A_1 = -\frac{\alpha+\beta}{(\alpha-\beta)^3}\,, \tag{2}$$

$$B_0 = \frac{\beta}{(\alpha-\beta)^2}\,, \qquad B_1 = \frac{\alpha+\beta}{(\alpha-\beta)^3}\,.$$

Putting, therefore, $x = h^{2n} z^2$, $\alpha = z_1^2$, and $\beta = z_2^2$, we shall have

$$\frac{h^{4n} z^4}{(h^{2n} z^2 - z_1^2)^2 (h^{2n} z^2 - z_2^2)^2} = \frac{z_1^2 \, h^{2n} z^2}{(z_1^2 - z_2^2)^2 (h^{2n} z^2 - z_1^2)^2} + \frac{z_2^2 \, h^{2n} z^2}{(z_1^2 - z_2^2)^2 (h^{2n} z^2 - z_2^2)^2} - \frac{(z_1^2 + z_2^2)\, h^{2n} z^2}{(z_1^2 - z_2^2)^3 (h^{2n} z^2 - z_1^2)} + \frac{(z_1^2 + z_2^2)\, h^{2n} z^2}{(z_1^2 - z_2^2)^3 (h^{2n} z^2 - z_2^2)}\,; \tag{3}$$

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* See Annals of Mathematics, Vol. VI, No. 4.

hence substituting in (1),

$$S_A(z) = C_1 \,\mathrm{p}\,(u-u_1) + C_2\,\mathrm{p}\,(u-u_2) + C_3 \frac{\sigma'}{\sigma}(u-u_1) + C_4 \frac{\sigma'}{\sigma}(u-u_2) + C_5$$

$$= \sum_{+\infty}^{-\infty} {}^{n} \frac{\pi \tan^4\alpha \,.\, z_1^2\, h^{2n} z^2}{(z_1^2 - z_2^2)^2\,(h^{2n} z^2 - z_1^2)^2} + \sum_{+\infty}^{-\infty} {}^{n} \frac{\pi \tan^4\alpha \,.\, z_2^2\, h^{2n} z^2}{(z_1^2 - z_2^2)^2\,(h^{2n} z^2 - z_2^2)^2} \tag{4}$$

$$+ \sum_{+\infty}^{-\infty} {}^{n} \frac{\pi \tan^4\alpha \,.\, (z_2^4 - z_1^4) \,.\, h^{2n} z^2)}{(z_1^2 - z_2^2)^3 (h^{2n} z^2 - z_1^2)(h^{2n} z^2 - z_2^2)} \,.$$

From § 9, (4) of Weierstrass and Schwarz's Elliptic Function Formulas (Edition of 1885), we have

$$\mathrm{p}\, u = -\frac{\eta}{\omega} - \left[\frac{\pi}{\omega}\right]^2 \left\{ \frac{z^2}{(z^2-1)^2} + \sum {}^{n} \frac{h^{2n} z^2}{(1 - h^{2n} z^2)^2} + \sum {}^{n} \frac{h^{2n} z^{-2}}{(1 - h^{2n} z^{-2})^2} \right\} . \tag{5}$$

We had found (ANNALS, Vol. VI, p. 96) that, when u becomes $u - u_1$ or $u - u_2$, z becomes $\frac{z}{z_1}$ or $\frac{z}{z_2}$ respectively.

Hence,

$$\mathrm{p}\,(u-u_1) = -\frac{\eta}{\omega} - \left[\frac{\pi}{\omega}\right]^2 \left\{ \frac{z^2 z_1^2}{(z^2 - z_1^2)^2} + \sum {}^{n} \frac{h^{2n} z^2 z_1^2}{(z_1^2 - h^{2n} z^2)^2} \right.$$

$$\left. + \sum {}^{n} \frac{h^{2n} z^{-2} z_1^{-2}}{(z_1^{-2} - h^{2n} z^{-2})^2} \right\}, \tag{6}$$

$$= -\frac{\eta}{\omega} - \left[\frac{\pi}{\omega}\right]^2 . \, z_1^2 \,.\, \sum_{+\infty}^{-\infty} {}^{n} \frac{h^{2n} z^2}{(h^{2n} z^2 - z_1^2)^2}, \tag{7}$$

$$\mathrm{p}\,(u-u_2) = -\frac{\eta}{\omega} - \left[\frac{\pi}{\omega}\right]^2 . \, z_2^2 \,.\, \sum_{+\infty}^{-\infty} {}^{n} \frac{h^{2n} z^2}{(h^{2n} z^2 - z_2^2)^2} . \tag{8}$$

From page 96, Volume VI of the ANNALS, just below formula (9), we have

$$\sum_{+\infty}^{-\infty} {}^{n} \frac{h^{2n} z^2}{(h^{2n} z^2 - z_1^2)\,(h^{2n} z^2 - z_2^2)} = \frac{\omega}{\pi i\,(z_1^2 - z_2^2)} \cdot \left\{ \frac{\sigma'}{\sigma}(u-u_1) - \frac{\sigma'}{\sigma}(u-u_2) \right\}$$

$$+ \frac{\eta}{\pi i} \frac{(u_1 - u_2)}{(z_1^2 - z_2^2)} ; \tag{9}$$

and from (7) and (8),

$$\sum_{+\infty}^{-\infty}{}_n \frac{h^{2n} z^2}{(h^{2n} z^2 - z_1^2)^2} = -\left[\frac{\omega}{\pi z_1}\right]^2 . \mathrm{p}(u - u_1) - \frac{\omega \eta}{(\pi z_1)^2}, \tag{10}$$

$$\sum_{+\infty}^{-\infty}{}_n \frac{h^{2n} z^2}{(h^{2n} z^2 - z_2^2)^2} = -\left[\frac{\omega}{\pi z_2}\right]^2 . \mathrm{p}(u - u_2) - \frac{\omega \eta}{(\pi z_2)^2}. \tag{11}$$

Substituting the values of the $\sum$'s in (9), (10), and (11) in (4), we have,

$$C_1 \mathrm{p}(u - u_1) + C_2 \mathrm{p}(u - u_2) + C_3 \frac{\sigma'}{\sigma}(u - u_1) + C_4 \frac{\sigma'}{\sigma}(u - u_2) + C_5$$

$$= -\frac{\omega^2}{\pi} \cdot \frac{\tan^4 a}{(z_1^2 - z_2^2)^2} \cdot \mathrm{p}(u - u_1) - \frac{\omega}{\pi} \cdot \frac{\eta \tan^4 a}{(z_1^2 - z_2^2)^2}$$

$$- \frac{\omega^2}{\pi} \cdot \frac{\tan^4 a}{(z_1^2 - z_2^2)^2} \cdot \mathrm{p}(u - u_2) - \frac{\omega}{\pi} \cdot \frac{\eta \tan^4 a}{(z_1^2 - z_2^2)^2}$$

$$- \frac{\omega \tan^4 a}{i} \cdot \frac{z_1^2 + z_2^2}{(z_1^2 - z_2^2)^3} \left\{ \frac{\sigma'}{\sigma}(u - u_1) - \frac{\sigma'}{\sigma}(u - u_2) \right\}$$

$$- \frac{(z_1^2 + z_2^2)}{(z_1^2 - z_2^2)^3} \cdot \frac{\eta}{i} \cdot \frac{\tan^4 a}{(u_1 - u_2)^{-1}}. \tag{12}$$

The C's, therefore, must have the following values:

$$C_1 = \quad C_2 = -\frac{\omega^2}{\pi} \cdot \frac{\tan^4 a}{(z_1^2 - z_2^2)^2}, \tag{13}$$

$$C_3 = - C_4 = - \frac{\omega \tan^4 a}{i} \cdot \frac{z_1^2 + z_2^2}{(z_1^2 - z_2^2)^3}, \tag{14}$$

$$C_5 = \frac{\eta \tan^4 a}{(z_1^2 - z_2^2)^2} \left\{ - \frac{2\omega}{\pi} - \frac{z_1^2 + z_2^2}{z_1^2 - z_2^2} \cdot \frac{u_1 - u_2}{i} \right\} \tag{15}$$

Our original problem is now completely solved, since $S_A(z)$ has been shown to depend upon p-function of $u - u_1$, $u - u_2$, and $\frac{\sigma'}{\sigma}$-function of the same, and upon the coefficients C_1, C_2, C_3, C_4, C_5, which involve z_1, z_2, u_1, u_2, and η, ω, a. How these latter elements may be calculated has already been shown in the ANNALS, Vol. VI, p. 97.

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